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ON THE NOTIONS OF DIMENSION AND TRANSCENDENCE DEGREE FOR MODELS OF ZFC

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Abstract. We define notions of generic dimension and generic transcendence degree between models of ZFC and give some examples.

1. introduction

Given models $V \subseteq W$ of ZFC, we define the notions of generic dimension and generic transcendence degree of W over V, and prove some results about them. We usually assume that W is a generic extension of V by a set or a class forcing notion, but some of our results work for general cases.

2. generic dimension and generic transcendence degree

Suppose that $V \subseteq W$ are models of ZFC with the same ordinals. Let's start with some definitions.

- Let $X = \langle x_i : i \in I \rangle \in W$, where I, the set of indices, is in V. The elements of X are called mutually generic over V, if for any partition $I = I_0 \cup I_1$ of I in V, $\langle x_i : i \in I_0 \rangle$ is generic over $V[\langle x_i : i \in I_1 \rangle]$, for some forcing notion $\mathbb{P} \in V$.
- The $\kappa-$generic transcendence degree of W over V, $\kappa - g.tr.deg_V(W)$, is defined to be $\sup\{|A| : A \in W$ and for all $X \in [A]^{<\kappa}$, the elements of X are mutually generic over $V\}$.
- Upward generic dimension of W over V, $[W : V]^U$, is defined to be $\sup\{\alpha :$ there exists a $\subset -$increasing chain $\langle V_i : i < \alpha \rangle$ of set generic extensions of V, where $V_0 = V$, and each $V_i \subseteq W\}$.

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- Downward generic dimension of W over V, $[W:V]_D$, is defined to be $\sup\{\alpha :$ there exists a $\subset -$decreasing chain $\langle W_i : i<\alpha\rangle$ of grounds [1] of W, where $W_0=W$, and each $W_i \supseteq V\}$.

Lemma 2.1. (1) $\kappa<\lambda \Rightarrow \kappa - g.tr.deg_V(W) \geq \lambda - g.tr.deg_V(W)$.

(2) $[W:V]^U \geq \kappa - g.tr.deg_V(W)$, *for* κ *such that* $\kappa - g.tr.deg_V(W) = \kappa^+ - g.tr.deg_V(W)$ *(such a* κ *exists by* (1)*).*

Lemma 2.2. [2] *Let* $V[G]$ *be a generic extension of* V *by some forcing notion* $\mathbb{P}\in V$, *and let* W *be a model of* ZFC *such that* $V\subseteq W\subseteq V[G]$. *Then there are* $\mathbb{Q}\in V$ *and* $\pi:\mathbb{P}\to\mathbb{Q}$ *such that:*

(1) $|\mathbb{Q}|\leq|\mathbb{P}|$,

(2) $\pi[G]$ *generate a filter* H *which is* $\mathbb{Q}-$*generic over* V,

(3) $W=V[H]$.

Proof. Let $\mathbb{B}=r.o(\mathbb{P})$, and let $e:\mathbb{P}\to\mathbb{B}$ be the induced embedding. Let $\bar{G}$ be the filter generated by $e[G]$, so that $V[G]=V[\bar{G}]$. Then for some complete subalgebra $\mathbb{C}$ of $\mathbb{B}$, we have $W=V[\bar{G}\cap\mathbb{C}]$. Let $\pi:\mathbb{B}\to\mathbb{C}$ be the standard projection map, given by $\pi(b)=\min\{c\in\mathbb{C}: c\geq b\}$. Let $\mathbb{Q}=\pi[\mathbb{P}]$, and consider $\pi\restriction\mathbb{P}:\mathbb{P}\to\mathbb{Q}$. $\mathbb{Q}$ is a dense subset of $\mathbb{C}$, and so it is easily seen that $\pi\restriction\mathbb{P}$ and $\mathbb{Q}$ are as required. $\square$

Corollary 2.3. *Let* $V[G]$ *be a generic extension of* V *by some forcing notion* $\mathbb{P}\in V$. *Then* $[V[G]:V]^U, [V[G]:V]_D \leq (2^{|\mathbb{P}|})^+$.

Question 2.4. *In the above Corollary, can we replace* $2^{|\mathbb{P}|}$ *by* $|\mathbb{P}|^{<\kappa}$, *where* κ *is such that* $\mathbb{P}$ *satisfies the* $\kappa-c.c.$?

Theorem 2.5. *Let* $\mathbb{P}=\mathrm{Add}(\omega,1)$ *be the Cohen forcing for adding a new Cohen real and let* G *be* $\mathbb{P}-$*generic over* V. *Then:*

(1) $\omega - g.tr.deg_V(V[G]) = 2^{\aleph_0}$.

(2) *For* $\kappa>\omega$ *we have* $\kappa - g.tr.deg_V(V[G])=\aleph_0$.

[1]Recall that V is a ground of W, if W is a set generic extension of V by some forcing notion in V.

[2]We thank Monroe Eskew for bringing this lemma to our attention.

(3) $[V[G]:V]^U = \aleph_1$.

(4) $[V[G]:V]_D = \aleph_1$.

In the proof of the above theorem, we will use the following.

Lemma 2.6. (1) *Forcing with* $\mathrm{Add}(\omega,1)$ *over* V*, can not add a generic sequence for* $\mathrm{Add}(\omega,\omega_1)$ *over* V*.*

(2) *A sequence* $\langle x_\alpha : \alpha < \aleph_1\rangle$ *of reals is* $\mathrm{Add}(\omega,\omega_1)$*-generic over* V*, iff for any countable set* $I \in V, I \subseteq \omega_1$*, the sequence* $\langle x_\alpha : \alpha \in I\rangle$ *is* $\mathrm{Add}(\omega,I)$*-generic over* V*, where* $\mathrm{Add}(\omega,I)$ *is the Cohen frcing for adding* I*-many Cohen reals indexed by* I*.*

(3) *If* $\mathbb{P}$ *is a non-trivial countable forcing notion, then* $\mathbb{P} \simeq \mathrm{Add}(\omega,1)$*.*

A generalized version of (1) is proved in [3], (2) follows easily using the fact that the forcing notion $\mathrm{Add}(\omega,\omega_1)$ satisfies the countable chain condition, and (3) is well-known.

Proof. (1) : In V, fix a canonical enumeration $F : 2^{<\omega} \to \omega$ such that if $|s| < |t|$, then $F(s) < F(t)$. For any $t \in (2^\omega)^V$, define g_t by $g_t(n) = g(F(t \restriction n))$. Then $\langle g_t : t \in (2^\omega)^V\rangle$ witnesses $\omega - g.tr.deg_V(V[G]) = 2^{\aleph_0}$.

(2) : Let $\kappa > \aleph_0$. As $\mathbb{P} \simeq \mathrm{Add}(\omega,\omega)$, it is clear that $\kappa - g.tr.deg_V(V[G]) \geq \aleph_0$. On the other hand, if $\kappa - g.tr.deg_V(V[G]) > \aleph_0$, then let $\langle x_\alpha : \alpha < \aleph_1\rangle \in V[G]$ be such that for any countable set $I \in V, I \subseteq \omega_1$, the sequence $\langle x_\alpha : \alpha \in I\rangle$ is a set of mutually generics over V. By Lemmas 2.2 and 2.6(3), each x_α can be viewed as a Cohen real, so it follows from Lemma 2.6(2) that the sequence $\langle x_\alpha : \alpha < \aleph_1\rangle$ is $\mathrm{Add}(\omega,\omega_1)$-generic over V, which contradicts Lemma 2.6(1).

(3) : Given any $\alpha < \aleph_1$, we have $\mathbb{P} \simeq \mathrm{Add}(\omega,\alpha)$, so let $\langle x_\beta : \beta < \alpha\rangle \in V[G]$ be $\mathrm{Add}(\omega,\alpha)$-generic over V, and define $V_\beta = V[\langle x_i : i < \beta\rangle]$, for $\beta \leq \alpha$ Then $V = V_0 \subset V_1 \subset \cdots \subset V_\alpha$ is an increasing chain of length α. So $[V[G]:V]^U \geq \alpha$.

Now assume on the contrary that, we have an increasing chain $V = V_0 \subset V_1 \subset \cdots \subset V_\alpha \cdots \subseteq V[G], \alpha < \aleph_1$, of generic extensions of V. For each $\alpha < \aleph_1$, set $V_\alpha = V[G_\alpha]$, where $G_\alpha \in V[G]$ is generic over some forcing notion in V. Again using Lemmas 2.2 an d2.6(3), we can assume that each $G_{\alpha+1}$ is some Cohen real $x_{\alpha+1}$ over V_α. Thus we have a sequence $\langle x_\alpha : \alpha < \aleph_1\rangle$ of reals, each x_α is Cohen generic over $V[\langle x_\beta : \beta < \alpha\rangle]$. By Lemma 2.6(2), the sequence $\langle x_\alpha : \alpha < \aleph_1\rangle$ is $\mathrm{Add}(\omega,\omega_1)$-generic over V, which contradicts Lemma 2.6(1).

(4) can be proved similarly. □

Using forcing notions producing minimal generic extensions, we can prove the following.

Theorem 2.7. *For any $0 < n < \omega$, there is a generic extension $V[G]$ of the V in which $[V[G] : V]^U = [V[G] : V]_D = n$.*

V. Kanovei noticed the following:

Theorem 2.8. *There is a generic extension $L[G]$ of L in which $[L[G] : L]^U = \omega + 1$.*

It follows from the results in [4] that:

Theorem 2.9. *Given any $\alpha \leq \omega_1$, there exists a cofinality preserving generic extension $L[G]$ of L in which $[L[G] : L]_D = \alpha + 1$.*

Question 2.10. *Given cardinals $\kappa_0 \geq \kappa_1 \geq ... \geq \kappa_n$, is there a forcing extension $V[G]$ of V, in which $\aleph_i - g.tr.deg_V(V[G]) = \kappa_i, i = 0, ..., n$ and $\lambda - g.tr.deg_V(V[G]) = \kappa_n$, for all $\lambda \geq \aleph_n$?*

Remark 2.11. *Force with $\mathbb{P} = \prod_{i=0}^{n} \mathrm{Add}(\aleph_i, \kappa_i)$, and let G be $\mathbb{P}$-generic over V. Then for all $0 \leq i \leq n, \aleph_i - g.tr.deg_V(V[G]) \geq \kappa_i$.*

Question 2.12. *Let $\alpha_0, \alpha_1 \leq \lambda$, where α_0, α_1 are ordinals and λ is a cardinal. Is there a generic extension $V[G]$ of V (possibly by a class forcing notion) in which $[V[G] : V]^U = \alpha_0, [V[G] : V]_D = \alpha_1$ and the number of intermediate submodels of V and $V[G]$ is λ?*

Using results of [1] and the fact that $0^\sharp$ can not be produced by set forcing, we have:

Theorem 2.13. *For all $\kappa, \kappa - g.tr.deg_L(L[0^\sharp]) = \infty$. Also we have $[L[0^\sharp] : L]^U = \infty$ and $[L[0^\sharp] : L]_D = 0$.*

3. Some non-absoluteness results

In this section we present some results which say that the notions of generic dimension and generic transcendence degree are not absolute between different models of ZFC.

Theorem 3.1. *There exists a generic extension V of L, such that if R is* $\mathrm{Add}(\omega, 1)$*-generic over V, then:*

(1) $\aleph_1 - g.tr.deg_V(V[R]) = \aleph_0$,

(2) $\aleph_1 - g.tr.deg_L(V[R]) \geq \aleph_1$.

Proof. By [2], there is a cofinality preserving generic extension V of L, such that adding a Cohen real R over V, adds a generic filter for $\mathrm{Add}(\omega, \omega_1)$ over L.

Now (1) follows from Theorem 2.5(2), and (2) follows from the fact that there exists a sequence $\langle x_\alpha : \alpha < \aleph_1 \rangle$ of reals, which is $\mathrm{Add}(\omega, \omega_1)$-generic over L. □

Theorem 3.2. *Assume $0^\sharp$ exists, κ is a regular cardinal in L, and let $\mathbb{P} = Sacks(\kappa, 1)_L$, the forcing for producing a minimal extension of L by adding a new subset of κ. Let G be $\mathbb{P}$-generic over V. Then:*

(1) $[L[G] : L]^U = [L[G] : L]_D = 1$,

(2) $[V[G] : V]^U, [V[G] : V]_D \geq \kappa$.

Proof. (1) is trivial, as forcing with $\mathbb{P}$ over L produces a minimal generic extension of L. (2) Follows from a result of Stanley [5], which says that forcing with $\mathbb{P}$ over V collapses κ into ω. □

References

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